

Some remarks on mirror symmetry and noncommutative elliptic curves

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Abstract

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1 Introduction

This paper starts from the question of how to extend the well known mirror symmetry of elliptic curves (see [Dij 1994]) - which was a motivating guide for the formulation of the homological mirror symmetry conjecture in [Kon] - to the noncommutative case. We do not claim to perform this extension, here, but restrict to a few simple remarks on the subject.

We start in the next section with two examples which motivate the consideration of noncommutative extensions of mirror symmetry. The second example even shows a case where such an extension is not an option but necessary. Besides this, the second example can - in a simple special case - directly be reduced to the question of a noncommutative extension of mirror symmetry for elliptic curves. In section 3, we collect some of the needed results on mirror symmetry of (commutative) elliptic curves and in section 4 we consider the case of noncommutative elliptic curves.

We will suggest a definition of Gromov-Witten invariants for noncommutative elliptic curves by a suitable generalization of a fermion partition function on elliptic curves. Besides this, we will argue that the bosonic version of this theory - which in the classical case corresponds to the Kodaira-Spencer theory formulation on the mirror - might correspond to a 12-dimensional theory with a cubic interaction term.

2 Motivation

Let us briefly discuss two examples which serve as motivation for the question of how mirror symmetry extends to the case of noncommutative manifolds.

The first example comes from the work of [KW]. There, the geometric structures and the statement of the geometric Langlands program for algebraic curves are derived from the S -duality conjecture for $N = 4$ SUSY YM-theory in $d = 4$ by reducing a topologically twisted form of the four dimensional gauge theory (with gauge group G) to a two dimensional sigma model with the target space given by the Hitchin moduli space $Hit(G, C)$ of the compact Riemann surface C - the compactification space in the dimensional reduction - and the gauge group G . The topologically twisted gauge theory contains a - so called canonical - parameter Ψ which combines the coupling parameter τ of the SUSY YM-theory

$$\tau = \frac{\theta}{2\pi} + \frac{4\pi i}{e^2}$$

(where θ is the θ -angle and e gives the gauge coupling) with the twisting parameter t (which parametrizes the family of possible topological twistings of the type used in [KW]) as

$$\Psi = \frac{\tau + \bar{\tau}}{2} + \frac{\tau - \bar{\tau}}{2} \left(\frac{t - t^{-1}}{t + t^{-1}} \right)$$

The two sides of the geometric Langlands duality arise at parameter values $\Psi = 0$ and $\Psi = \infty$, respectively.

One can now pose the question of what happens at a more general parameter value $\Psi \in \mathbb{C}$. For $\Psi \in \mathbb{R}$ one may always assume $t = 1$ and hence

$$\Psi = \frac{\theta}{2\pi} = \text{Re}(\tau)$$

With ω_I denoting the symplectic form corresponding to the I choice of complex structure on the Hyperkähler manifold $Hit(G, C)$ (see [KW] for the details), the 2d sigma model in this case contains a B -field

$$B = -\omega_I \operatorname{Re}(\tau)$$

So, for more general values of Ψ a B -field arises and one gets a twisted version of geometric Langlands duality (quantum geometric Langlands duality, see section 11.3 of [KW]). Since the S -duality of the 4d gauge theory reduces to (homological) mirror symmetry for the 2d sigma model, the question for a noncommutative extension of mirror symmetry arises, here.

As a second example, let us consider the case of the $D5$ -brane world-volume gauge theory in type IIB string theory on a - possibly singular - $K3$ -surface X (or $X = T^4$). This theory appears as the effective limit of little string theory (LST) in type IIB. One can show that the gauge group of the LST - and of the 6d effective field theory - has to belong to one of the three ADE -series. We will completely restrict to the $U(1)$ -case in the A -series and to $X = T^4$, here.

Concretely, let M be the worldvolume of a $D5$ -brane in type IIB string theory, F the curvature of a $U(1)$ -connection of a line bundle over M , B the NS 2-form field. Let \mathcal{C} be the background RR gauge field, i. e.

$$\mathcal{C} = \theta + \tilde{B} + G$$

with θ a scalar, \tilde{B} the RR 2-form field and G a 4-form field with self-dual field strength

$$dG = *dG$$

Let v be the RR charge vector (Mukai vector)

$$v = \operatorname{Tr} \exp \left(\frac{iF}{2\pi} + B + \frac{c_2}{24} \right)$$

With

$$\mathcal{F} = F - 2\pi i B$$

the action of the effective 6d CFT can be written as (see [Dij 1998] for more details)

$$S = \int_M \frac{1}{g_s} \operatorname{Tr} \mathcal{F} \wedge * \mathcal{F} + \mathcal{C} \wedge v(\mathcal{F}) \quad (1)$$

Consider now the case

$$M = \Sigma \times X$$

with Σ a Riemann surface (or non-compact) and $X = T^4$ with

$$\text{vol}(X) \ll \text{vol}(\Sigma)$$

In the limit of small X , (1) can, again, be dimensionally reduced to a sigma model on Σ with the target space given by $\text{Inst}(X)$, the instanton moduli space (anti-self-dual connections $F_+ = 0$) on X . For X Hyperkähler (i.e. $X = T^4$ or X a K3 surface), $\text{Inst}(X)$ is a - singular - Hyperkähler manifold.

In the $U(1)$ -case, the singular Hyperkähler structure of $\text{Inst}(X)$ can be regularized to a smooth Hyperkähler manifold by the large N limit of the Hilbert scheme of N points on X . On the other hand, the Hilbert scheme of N points on X also regularizes the orbifold $S^N X$, the N -fold power of X modulo the action of the symmetric group.

Let us pose the question if there exists a canonical coisotropic brane (c.c. brane), i.e. a target space filling coisotropic brane, for this Hyperkähler sigma model. In the case of the Hitchin moduli space sigma model, the c.c. brane is used in [KW] to derive the D -module property of the Hecke eigensheaves, i.e. it is essential to derive the structures of the geometric Langlands program from S -duality (also the existence of the c.c. brane is believed - [Wit] - to be essential to derive the 2d conformal field theory approach to the geometric Langlands program - see [Fre] - from the setting of [KW]).

In [Dij 1998] it was shown that the RR -fields of (1) contribute to the NS 2-form field of the $\text{Inst}(X)$ sigma model under dimensional reduction while both the Kähler form and the NS 2-form B -field on X in (1) are used to determine the Kähler form on $\text{Inst}(X)$, i.e. the Hyperkähler structure on $\text{Inst}(X)$ is not well defined unless one specifies the B -field on X .

The condition for the existence of a coisotropic A -brane of full dimension (c.c. brane) in a sigma model with target Y was shown in [KO] to be:

$$\left(\omega^{-1}\tilde{F}\right)^2 = -1 \tag{2}$$

with \tilde{F} the curvature of the connection of the bundle defining the c.c. brane and ω the symplectic form on Y .

Assume, now, that the NS 2-form field B in (1) vanishes. In this case (see [Dij 1998]), the Kähler structure of $\text{Inst}(X)$ is given by the large N

limit of $S^N X$, i.e. the Kähler structure is determined by the symmetric N -fold products of the Kähler structure of X . But for $X = T^4$ and generic ω , it was argued in [KO] that a coisotropic brane of full dimension should be impossible. This argument can be adapted to $S^N X$. This means that generically there should exist no c.c. brane on $Inst(X)$ and the sigma model defined by the dimensional reduction of (1) should therefore radically differ in this respect from the setting of [KW].

Up to now, we have implicitly assumed that the RR background gauge fields of (1) vanish, i.e. the $Inst(X)$ sigma model has a vanishing NS 2-form field \widehat{B} . Let us now assume that $\widehat{B} \neq 0$. But we have

$$d\widehat{B} = 0$$

We consider the large volume limit of $Inst(X)$ (corresponding to small string coupling g_s , see [Dij 1998]).

Let $\widetilde{\mathcal{F}}$ be the curvature \widetilde{F} shifted by \widehat{B} , i.e.

$$\widetilde{\mathcal{F}} = \widetilde{F} - 2\pi i \widehat{B}$$

Equation (2) should then be replaced by

$$\left(\omega^{-1} \widetilde{\mathcal{F}}\right)^2 = -1 \tag{3}$$

It can be shown that the field \widehat{B} can always be fine-tuned such that a c.c. brane exists (i.e. the RR background field can always be fine-tuned to allow for the existence of a c.c. brane). Here, one makes use of the fact that the components of the NS 2-form field of the $Inst(X)$ sigma model induced from the RR background gauge fields of (1) constitute a basis of $H^2(Inst(X), \mathbb{R})$ (see [Dij 1998]).

Actually, the components of \widehat{B} induced from θ and \widetilde{B} already constitute a basis of $H^2(Inst(X), \mathbb{R})$ while we have the following additional condition on the RR background gauge fields (see [Dij 1998]): If

$$v \in (Q_5, Q_3, -Q_1) \in H^*(X, \mathbb{Z})$$

is the Mukai vector, we have

$$v \cdot \mathcal{C} = Q_1 \cdot \theta + Q_3 \cdot \widetilde{B} - Q_5 \cdot G = 0$$

So, if G would vanish, we would have an additional relation between θ and \tilde{B} , violating the basis property of the \hat{B} -field contributions induced by them. In consequence, G can not vanish.

We can therefore draw the following conclusion: For non-vanishing RR background 4-form field G in (1) - and therefore for non-vanishing NS 2-form field \hat{B} in the $Inst(X)$ sigma model - there exists always a c.c. brane in the $Inst(X)$ sigma model.

In consequence, if we want to study mirror symmetry in the sigma model reduction of (1) in the presence of a c.c. brane, we once again arrive at the question of a noncommutative extension of mirror symmetry. Observe that in this case the noncommutative extension is not just an option for a generalization but is necessary since for $\hat{B} = 0$ a c.c. brane does not exist generically.

As a special case, one can show that one can choose a \hat{B} which is induced from a 2-form field on $X = T^4$ on the symmetric powers $S^N X$. We can make an even more special choice by requiring that the 2-form field on T^4 should respect the factorization

$$T^4 \cong T^2 \times T^2$$

and be constant. In consequence, we can study the effect of non-vanishing 4-form field G in (1) in a special case by studying a field \hat{B} which is induced from a simple constant 2-form field on an elliptic curve.

In other words, in the simple case of such a factorizable field \hat{B} , we can study the question of a noncommutative extension of mirror symmetry in the sigma model reduction of (1) by starting from the question of a noncommutative extension of mirror symmetry for elliptic curves. It is this question which forms the topic of the present paper. We do not claim to present a noncommutative extension of mirror symmetry for elliptic curves, here, but restrict to a few small remarks on the subject.

3 The elliptic curve

Let us start by very briefly reviewing the case of mirror symmetry for (commutative) elliptic curves (see [Dij 1994] and references cited therein, especially [Dou], [KaZa], [Rud]).

An elliptic curve $E_{t,\tau}$ is a smooth 2-torus equipped with a holomorphic and a symplectic structure. The holomorphic structure - parametrized by τ

- is given by the representation of the elliptic curve as

$$\mathbb{C}/(\mathbb{Z} \oplus \mathbb{Z}\tau)$$

with $\tau = \tau_1 + i\tau_2 \in \mathbb{C}$ from the upper half plane \mathbb{H} , i.e. $\tau_2 > 0$. The symplectic structure - parametrized by $t \in \mathbb{H}$ - is given by the complexified Kähler class $[\omega] \in H^2(E_{t,\tau}, \mathbb{C})$ with

$$\omega = -\frac{\pi t}{\tau_2} dz \wedge d\bar{z}$$

and for $t = t_1 + it_2$ the area of the elliptic curve is given by t_2 . Mirror symmetry relates the elliptic curves $E_{t,\tau}$ and $E_{\tau,t}$.

On the symplectic side, we have the Gromov-Witten invariants F_g , defined as the generating functions for counting d -fold connected covers of $E_{t,\tau}$ in genus g . One can combine the functions F_g into a two-variable partition function

$$Z(q, \lambda) = \exp \sum_{g=1}^{\infty} \lambda^{2g-2} F_g(q)$$

with $q = e^{2\pi i t}$.

Now, it is important that $Z(q, \lambda)$ can be calculated in three different ways (see Theorem 1 - Theorem 3 of [Dij 1994]). The first case is a large N calculation in terms of $U(N)$ Yang-Mills theory on $E_{t,\tau}$. We will not refer to this case, here. The second possibility is a calculation in terms of a Dirac fermion on the elliptic curve. Starting from Dirac spinors b, c on the elliptic curve with action

$$S = \int_{E_{t,\tau}} (b\bar{\partial}c + \lambda b\partial^2 c)$$

one shows that the operator product expansion defines a fermionic representation of the $W_{1+\infty}$ algebra. The partition function can be calculated as a generalized trace (as defined in [AFMO]) of this algebra, leading to

$$Z(q, \lambda) = q^{-\frac{1}{24}} \oint \frac{dz}{2\pi i z} \prod_{p \in \mathbb{Z}_{\geq 0} + \frac{1}{2}} \left(1 + zq^p e^{\lambda p^2}\right) \left(1 + \frac{1}{z} q^p e^{-\lambda p^2}\right) \quad (4)$$

(see [Dij 1996] for the details). Note that for the action and the partition function above - and for the sequel of this paper - we have changed the notation to denote the parameter values of the mirror elliptic curve by t and

τ . It is this representation of $Z(q, \lambda)$ which leads to the famous theorem of Dijkgraaf, Kaneko, Zagier stating that the functions $F_g(q)$ are quasi-modular forms (i.e. $F_g \in \mathbb{Q}[E_2, E_4, E_6]$ where E_2, E_4, E_6 are the classical Eisenstein series of weight 2, 4, and 6, respectively) and have weight $6g - 6$.

Finally, as in the case of Calabi-Yau 3-folds, by mirror symmetry $Z(q, \lambda)$ can be calculated as the partition function of a Kodaira-Spencer theory. In the case of elliptic curves, this is given by the action of a simple real bosonic field with $(\partial\varphi)^3$ interaction term, i.e. by the action

$$S(\varphi) = \int_{E_{\tau,t}} \left(\frac{1}{2} \partial\varphi \bar{\partial}\varphi + \frac{\lambda}{6} (-i\partial\varphi)^3 \right)$$

(see [Dij 1994], [Dij 1996] for the details).

4 The noncommutative elliptic curve

Let us now come to the question how mirror symmetry and the above results might generalize to the noncommutative torus. We will start with the case of the fermionic representation of $Z(q, \lambda)$. The first question we have to face is how the full structure of an elliptic curve, beyond the structure of a smooth torus, generalizes to the noncommutative case. Holomorphic structures on the noncommutative torus have been introduced in [Pol 2003], [Pol 2004], [Pol 2005], and [PS]. Unfortunately, $Z(q, \lambda)$ is not expressed in terms of a single elliptic curve but in terms of the modular parameter q of the whole family of elliptic curves. So, to arrive at an analogue of (4), we have to consider an extension of the range of the modular parameter, including noncommutative elliptic curves. In [Soi] it is shown that one can view the noncommutative torus as the degenerate limit $|q| \rightarrow 1$ of classical elliptic curves (observe that since $t \in \mathbb{H}$ and $q = e^{2\pi it}$, $|q| < 1$ for classical elliptic curves), arriving in this way also at the notion of a noncommutative elliptic curve. We will therefore discuss the question of a noncommutative analogue of (4) in the form of the question of performing the limit $|q| \rightarrow 1$ in (4).

Obviously, we can not directly perform the limit in (4). Besides this, there exist only very few results on q -analysis for $|q| = 1$. But fortunately there exists an elliptic deformation of the q -deformed gamma function and this elliptic gamma function (which has two deformation parameters) allows to take a limit in which a single unimodular deformation parameter arises

([Rui 1997], [Rui 2001]). In this sense, the elliptic gamma function includes the q -gamma function case with $|q| = 1$. We will therefore consider the problem of taking the limit $|q| \rightarrow 1$ in (4) in the more general form of looking for an elliptic analogue of (4). We will proceed as follows: We will first rewrite (4) in terms of q -deformed gamma functions (for the classical case, i.e. $|q| < 1$) and then replace these by the elliptic gamma function of [Rui 1997], [Rui 2001].

Let us start by considering the case $\lambda = 0$. With the substitution $q \mapsto q^2$, we have

$$\begin{aligned} Z(q, 0) &= q^{-\frac{1}{12}} \oint \frac{dz}{2\pi i z} \prod_{j \geq 0} (1 + zq^{2j+1}) \left(1 + \frac{1}{z}q^{2j+1}\right) \\ &= q^{-\frac{1}{12}} \oint \frac{dz}{2\pi i z} (-zq; q^2)_\infty \left(-\frac{q}{z}; q^2\right)_\infty \end{aligned}$$

where

$$(a; q)_n = \prod_{j=0}^{n-1} (1 - aq^j)$$

is the q -shifted factorial and

$$(a; q)_\infty = \prod_{j=0}^{\infty} (1 - aq^j)$$

the limit $n \rightarrow \infty$ which exists for $|q| < 1$. Remember that the classical Jacobi theta function

$$\vartheta(z, q) = \sum_{n=-\infty}^{n=+\infty} z^n q^{n^2}$$

can be expressed in the form of the Jacobi triple product as

$$\vartheta(z, q) = (-zq; q^2)_\infty \left(-\frac{q}{z}; q^2\right)_\infty (q^2; q^2)_\infty$$

i.e. $Z(q, 0)$ is basically given by an integral over the first two factors of $\vartheta(z, q)$.

Next, let us rewrite $(a; q)_\infty$ in terms of the function Γ_q with

$$\Gamma_q(x) = \frac{q^{-\frac{x^2}{16}}}{\left(-q^{\frac{1}{2}(x+1)}; q\right)_\infty}$$

Observe that this form of the q -deformed gamma function (as used e.g. in [Sto]) differs slightly from the usually used q -gamma function

$$\gamma_q(x) = \frac{(q; q)_\infty}{(q^x; q)_\infty} (1 - q)^{1-x}$$

Solving

$$a = -q^{\frac{1}{2}(x+1)}$$

for x , we arrive at

$$x = \frac{\log(a^2)}{\log(q)} - 1 = 2 \frac{\log(a)}{\log(q)} - 1$$

and

$$(a; q)_\infty = \frac{q^{-\left(\frac{2 \frac{\log(a)}{\log(q)} - 1}{4}\right)^2}}{\Gamma_q\left(2 \frac{\log(a)}{\log(q)} - 1\right)}$$

In consequence, we have

$$Z(q, 0) = q^{-\frac{1}{12}} \oint \frac{dz}{2\pi i z} \frac{q^{-\left(\frac{\log(-z)}{2 \log(q)}\right)^2}}{\Gamma_{q^2}\left(\frac{\log(-z)}{\log(q)}\right) \Gamma_{q^2}\left(-\frac{\log(-z)}{\log(q)}\right)} \quad (5)$$

Let for $q, p \in \mathbb{C}$ with $|q|, |p| < 1$

$$\Gamma(z; q, p) = \prod_{j,k=0}^{\infty} \frac{1 - z^{-1} q^{j+1} p^{k+1}}{1 - z q^j p^k} \quad (6)$$

be the elliptic gamma function of [Rui 1997], [Rui 2001]. Then an elliptic generalization of (5) - which allows to take the limit to unimodular q in (5) - would be

$$\begin{aligned} & Z(q, p, 0) \\ &= q^{-\frac{1}{12}} p^{-\frac{1}{12}} \oint \frac{dz}{2\pi i z} \frac{q^{-\left(\frac{\log(-z)}{2(\log(q)+\log(p))}\right)^2} p^{-\left(\frac{\log(-z)}{2(\log(q)+\log(p))}\right)^2}}{\Gamma\left(\frac{\log(-z)}{\log(q)+\log(p)}; q^2, p^2\right) \Gamma\left(-\frac{\log(-z)}{\log(q)+\log(p)}; q^2, p^2\right)} \end{aligned} \quad (7)$$

Observe that $\Gamma(z; q, p)$ is symmetric in q and p which guides our guess for the generalization of $Z(q, 0)$.

Let us now discuss the case $\lambda \neq 0$. The factor

$$(-zq; q^2)_\infty = \prod_{j=0}^{\infty} (1 + zq^{2j+1})$$

in $Z(q, 0)$ is in this case deformed to

$$\prod_{j=0}^{\infty} \left(1 + zq^{2j+1} e^{\frac{\lambda}{2}(j+\frac{1}{2})^2} \right)$$

Similarly, the factor

$$\left(-\frac{q}{z}; q^2\right)_\infty = \prod_{j=0}^{\infty} \left(1 + \frac{1}{z} q^{2j+1} \right)$$

is deformed to

$$\prod_{j=0}^{\infty} \left(1 + \frac{1}{z} q^{2j+1} e^{-\frac{\lambda}{2}(j+\frac{1}{2})^2} \right)$$

Since

$$\prod_{j=0}^{\infty} \frac{1 - z^{-1} q^{j+1}}{1 - zq^j} = \frac{(z^{-1}q; q)_\infty}{(z; q)_\infty}$$

we make the following Ansatz for a generalization of the elliptic gamma function (see (6)) to $\lambda \neq 0$:

$$\Gamma(z; q, p, \lambda) = \prod_{j,k=0}^{\infty} \frac{1 - z^{-1} q^{j+1} p^{k+1} e^{-\frac{\lambda}{2}(j+\frac{1}{2})^2} e^{-\frac{\lambda}{2}(k+\frac{1}{2})^2}}{1 - zq^j p^k e^{\frac{\lambda}{2}(j+\frac{1}{2})^2} e^{\frac{\lambda}{2}(k+\frac{1}{2})^2}}$$

Let

$$\alpha_{q,p,\lambda}(z) = \frac{\log(-z)}{\log(q) + \log(p) + \frac{9}{4}\lambda}$$

and

$$\widehat{\vartheta}(z; q, p, \lambda) = \frac{q^{-\frac{\alpha_{q,p,\lambda}^2(z)}{4}} p^{-\frac{\alpha_{q,p,\lambda}^2(z)}{4}} e^{-\frac{9}{16}\lambda\alpha_{q,p,\lambda}^2(z)}}{\Gamma(\alpha_{q,p,\lambda}(z); q^2, p^2, \lambda) \Gamma(-\alpha_{q,p,\lambda}(z); q^2, p^2, \lambda)}$$

With these definitions at hand, we make the following Ansatz for a generalization of $Z(q, p, 0)$ to $\lambda \neq 0$:

$$Z(q, p, \lambda) = q^{-\frac{1}{12}} p^{-\frac{1}{12}} \oint \frac{dz}{2\pi i z} \widehat{\vartheta}(z; q, p, \lambda) \quad (8)$$

Of course, the numerical factors in the definition of $\widehat{\vartheta}(z; q, p, \lambda)$ are in no way unique. We have chosen a definition, where the factors correspond to those appearing for $j = 1$ in the deformation of the q -shifted factorials appearing for $\lambda \neq 0$ (since this is how q and p appear in the $\lambda = 0$ case in the shifted factorials). We suggest (8) as the definition for the partition function of an elliptic fermion on the elliptic curve which contains the degeneration to a single unimodular parameter (corresponding to a fermion partition function on the noncommutative elliptic curve) as a special case.

Remark 1 *It is an open question for future research if (8) corresponds for a noncommutative elliptic curve to a fermionic action analogous to the action*

$$S = \int_{E_{t,\tau}} (b\bar{\partial}c + \lambda b\partial^2 c)$$

of the commutative case.

As in the classical case of commutative elliptic curves, we can use the partition function (8) to define Gromov-Witten invariants. Concretely, in the classical case the definition of the partition function as

$$Z(q, \lambda) = \exp \left(\sum_{g=1}^{\infty} \lambda^{2g-2} F_g(q) \right)$$

implies that we can calculate the Gromov-Witten invariants F_g as

$$F_g = \frac{1}{(2g-2)!} \frac{\partial^{2g-2} \log(Z)}{\partial \lambda^{2g-2}};_{\lambda=0} \quad (9)$$

We can now use (9), applied to the partition function (8) as a definition of elliptic Gromov-Witten invariants $F_g(q, p)$. The limit to a single unimodular parameter can be used as a definition of Gromov-Witten invariants for noncommutative elliptic curves.

Let us next consider the question of a noncommutative analogue of the bosonic $(\partial\varphi)^3$ -action. We do not have definitive results for this case but want to conclude this section with a few remarks. The bosonic action has two properties which are decisive for the calculation of Gromov-Witten invariants:

- The interaction term is cubic.
- The interaction term is chiral.

Let us assume that mirror symmetry extends to the noncommutative case. More concretely, let us assume that the partition function (8) has a representation by a bosonic action and that this action has (at least as one contribution) a cubic chiral interaction term.

In the classical case, the bosonic representation is given by a real boson, i.e. we have a real valued scalar field or more generally a section of a line bundle. In the case of (8), the integrand is mainly given by a product of elliptic gamma functions. Now, it has been shown in [FHRZ], [FV] that the elliptic gamma function is not related to a section of a line bundle but to a section of a gerbe. One might therefore suspect that a bosonic representation of (8) - if it exists - should also be given in terms of a bosonic field on a gerbe. So, one might suspect that locally the bosonic field φ is not given as a scalar but transforms as a 1-form (remember that all fields transforming locally as a p -form are bosonic). $\partial\varphi$ should then be replaced by the differential on forms, i.e. $\partial\varphi$ should locally transform as a 2-form. The interaction term would then be (in order to be cubic)

$$\partial\varphi \wedge \partial\varphi \wedge \partial\varphi$$

and hence a 6-form. Since the interaction term should be chiral, we should actually count degrees of forms in complex cohomology. Let us assume e.g. that the interaction term transforms locally as a $(0,6)$ -form. Analogous to the case of the $(0,3)$ -form in six dimensional Kodaira-Spencer theory, we should integrate this together with a $(6,0)$ -form. In consequence, we arrive at the conclusion that the bosonic theory - if it exists - should live on a 12-dimensional manifold. So, we are lead to pose the following questions:

- Does there exist a cubic twelve dimensional theory with the partition function given by (8)?

- If yes, how is the twelve dimensional manifold determined?
- In six dimensional Kodaira-Spencer theory the field is an element of a cohomology class, taking into account the gauge freedom of the field. One would expect something similar to happen for the 2-form field $\partial\varphi$. What is the correct type of cohomology theory?

We plan to come back to some of these questions in future work.

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